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M.Sc. (CBCS) DEGREE EXAMINATION, NOVEMBER 2022.

Third Semester

Mathematics — Core

MEASURE AND INTEGRATION

(For those who joined in July 2021 onwards)

Time : Three hours Maximum : 75 marks

PART A — (10 × 1 = 10 marks)

Answer ALL questions.

Choose the correct answer :

1. Let A be the set of irrational numbers in [0, 1]. Then $m^*(A)$ is

(a) ∞ (b) 0

(c) 1 (d) 1/2

2. If A is a measurable set of finite outer measure that is contained in B then $m^*(B \sim A) = m^*(B) - m^*(A)$. This property is known as

(a) finite subadditivity (b) translation invariant

(c) countably additive (d) excision property

3. For a function of defined on E, $f^+(x)$ is defined by

(a) $\max\{f(x) - f(x)\}$ (b) $\max\{f(x), 0\}$

(c) $\max\{-f(x), 0\}$ (d) $\min\{f(x), 0\}$

4. If for each $\varepsilon > 0$, then is an index N for which $|f - f_n| < \varepsilon$ on A for all $n \geq N$, we say that

(a) $\{f_n\}$ converges to f uniformly on A

(b) $\{f_n\}$ converges to f pointwise a.e. on A

(c) $\{f_n\}$ converges to f pointwise on A

(d) $\{f_n\} \to f$ as $n \to \infty$

5. Let f be a nonnegative measurable function on E. then for any $\lambda > 0$, $m\{x \in E / f(x) \geq \lambda\}$ is

(a) $= \lambda \int_E f$ (b) $\leq \lambda \int_E f$

(c) $\geq \lambda \int_E f$ (d) $= \frac{1}{\lambda} \int_E f$

6. Define $g_n = \chi_{(n,n+1)}$. Then $\lim_{n\to\infty} \int_{\mathbb{R}} g_n$ is

(a) 0 (b) 1

(c) ∞ (d) n

7. $[-5f]^+$ is

(a) $-5f^+$ (b) $5f^-$

(c) $5f^+$ (d) $-5f^-$

8. For all $a \leq u \leq v \leq b$, $\int_u^v Diff_h f$ is

(a) 0 (b) $Av_h f(v) + Av_h f(u)$

(c) $Av_h f(v) - Av_h f(u)$ (d) $Av_h f(u) - Av_h f(v)$

9. A function f on $[a, b]$ is absolutely continuous on $[a, b]$ if and only if

(a) it is an indefinite integral over $[a, a]$

(b) $\int_a^b f' = f(a) - f(a)$

(c) f is differentiable almost everywhere on (a, a)

(d) f is differentiable

10. If $\{A, B\}$ is a Hahn decomposition for r, then r^+ and r^- are defined by

(a) $r^+(E) = r(E \cap A)$, $r^-(E) = -r(E \cap B)$

(b) $r^+(E) = -r(E \cap A)$, $r^-(E) = r(E \cap B)$

(c) $r^+(E) = r(E \cup A)$, $r^-(E) = -r(E \cup B)$

(d) $r^+(E) = -r(E \cup A)$, $r^-(E) = r(E \cup B)$

PART B — (5 × 5 = 25 marks)

Answer ALL questions, choosing either (a) or (b).

11. (a) Define a measurable set and prove that any set of outer measure zero is measurable.

Or

(b) If m is the Lebesgue measure and if $\{A_K\}_{k=1}^{\infty}$ is an ascending collection of measurable sets, prove that $m\left(\bigcup_{k=1}^{\infty} A_k\right) = \lim_{k\to\infty} m(A_k)$.

12. (a) Let f be an extended real - valued function on E. If f is measurable on E and $f = g$ a.e. on E, prove that g is measurable on E.

Or

(b) Let $\{f_n\}$ be a sequence of measurable function on E that converges pointwise a.e. on E to the function f. Prove that f is measurable.

[P.T.O.]

13. (a) Let f be a bounded measurable function on a set of finite measure E. Prove that f is integrable over E.

Or

(b) State and prove Chebychev's inequality.

14. (a) Let f be integrable over E and $\{E_n\}_{n=1}^{\infty}$ a disjoint countable collection of measurable subsets of E whose union is E. Prove that $\int_E f = \sum_{n=1}^{\infty} \int_{E_n} f$.

Or

(b) Define the total variation $TV(f)$ of f on $[a,b]$. If f is a Lipschitz function on $[a,b]$, prove that f is of bounded variation of $[a,b]$.

15. (a) Define absolutely continuous functions. Prove that absolutely continuous functions are continuous. Is the converse true? Justify.

Or

(b) State and prove the Jordan decomposition theorem.

PART C — (5 × 8 = 40 marks)

Answer ALL questions, choosing either (a) or (b)

16. (a) Prove that the outer measure of an interval is its length.

Or

(b) (i) State and prove that Borel - Cantelli lemma.

(ii) State the continuity properties of Lebesque measure.

17. (a) Let f and g be measurable functions on E that are finite a.e. on E. For any α and β, prove that $\alpha f + \beta g$ is measurable on E and that fg is measurable on E.

Or

(b) State and prove Egoroff's theorem.

18. (a) State and prove the bounded convergence theorem.

Or

(b) Let f and g be nonnegative measurable functions on E. For any $\alpha > 0$ and $\beta > 0$, prove that $\int_E (\alpha f + \beta g) = \alpha \int_E f + \beta \int_E g$.

19. (a) State and prove the Lebesgue dominated convergence theorem.

Or

(b) State and prove the Vitali covering lemma.

20. (a) Let the function f be absolutely continuous on $[a,b]$. Prove that f is the difference of increasing absolutely continuous functions and in particular, is of bounded variation.

Or

(b) Let γ be a signed measure on the measurable space (X,M) and E a measurable set for which $0 < \gamma(E) < \infty$. Prove that there is a measurable subset A of E that is positive and of positive measure.

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